

# CATEGORIFICATION OF SIGN-SKEW-SYMMETRIC CLUSTER ALGEBRAS AND SOME CONJECTURES ON $\mathbf{g}$ -VECTORS

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**ABSTRACT.** Using the unfolding method given in [12], we prove the conjectures on sign-coherence and a recurrence formula respectively of  $\mathbf{g}$ -vectors for acyclic sign-skew-symmetric cluster algebras. As a following consequence, the conjecture is affirmed in the same case which states that the  $\mathbf{g}$ -vectors of any cluster form a basis of  $\mathbb{Z}^n$ . Also, the additive categorification of an acyclic sign-skew-symmetric cluster algebra  $\mathcal{A}(\Sigma)$  is given, which is realized as  $(\mathcal{C}^Q, \Gamma)$  for a Frobenius 2-Calabi-Yau category  $\mathcal{C}^Q$  constructed from an unfolding  $(Q, \Gamma)$  of the acyclic exchange matrix  $B$  of  $\mathcal{A}(\Sigma)$ .

## 1. INTRODUCTION

Cluster category was introduced by [2] for an acyclic quiver. In general, we view a Hom-finite 2-Calabi-Yau triangulated category  $\mathcal{C}$  which has a cluster structure as a cluster category, see [1]. In fact, the mutation of a cluster-tilting object  $T$  in  $\mathcal{C}$  categorifies the mutation of a quiver  $Q$ , where the quiver  $Q$  is the Gabriel quiver of the algebra  $\text{End}_{\mathcal{C}}(T)$ . Cluster character gives an explicit correspondence between certain cluster objects of  $\mathcal{C}$  and all the clusters of  $\mathcal{A}(\Sigma(Q))$ , where  $\Sigma(Q)$  means the seed associated with  $Q$ . For details, see [10], [14] and [15]. Thus, cluster category and cluster character are useful tools to study a cluster algebra.

Let  $\mathcal{A}(\Sigma)$  be a cluster algebra with principal coefficients at  $\Sigma = (X, Y, B)$ , where  $B$  is an  $n \times n$  sign-skew-symmetric integer matrix,  $Y = (y_1, \dots, y_n)$ . The celebrated Laurent phenomenon says that  $\mathcal{A}$  is a subalgebra of  $\mathbb{Z}[y_{n+1}, \dots, y_{2n}][X^{\pm 1}]$ . Setting  $\deg(x_i) = e_i$ ,  $\deg(y_j) = -b_j$ , then  $\mathbb{Z}[y_{n+1}, \dots, y_{2n}][X^{\pm 1}]$  becomes to a graded algebra, where  $\{e_i \mid i = 1, \dots, n\}$  is the standard basis of  $\mathbb{Z}^n$  and  $b_i$  is the  $i$ -th column of  $B$ . Under such  $\mathbb{Z}^n$ -grading, the cluster algebra  $\mathcal{A}$  is a graded subalgebra, in which the degree of a homogenous element is called its  **$\mathbf{g}$ -vector**; furthermore, each cluster variable  $x$  in  $\mathcal{A}(\Sigma)$  is homogenous, denoted its  $\mathbf{g}$ -vector as  $\mathbf{g}(x)$ . For details, see [9].

It was conjectured that

**Conjecture 1.1.** ([9], Conjecture 6.13) *For any cluster  $X'$  of  $\mathcal{A}(\Sigma)$  and all  $x \in X'$ , the vectors  $\mathbf{g}(x)$  are sign-coherent, which means that the  $i$ -th coordinates of all these vectors are either all non-negative or all non-positive.*

Such conjecture has been proved in the skew-symmetrizable case in ([11], Theorem 5.11).

**Conjecture 1.2.** ([9], Conjecture 7.10(2)) *For any cluster  $X'$  of  $\mathcal{A}(\Sigma)$ , the vectors  $\mathbf{g}(x), x \in X'$  form a  $\mathbb{Z}$ -basis of the lattice  $\mathbb{Z}^n$ .*

In terms of cluster pattern, fixed a regular tree  $\mathbb{T}_n$ , for any vertices  $t$  and  $t_0$  of  $\mathbb{T}_n$ , let  $\mathbf{g}_{1,t}^{B^0; t_0}, \dots, \mathbf{g}_{n,t}^{B^0; t_0}$  denote the  $\mathbf{g}$ -vectors of the cluster variables in the seed  $\Sigma_t$  with respect to the principal coefficients seed  $\Sigma_{t_0} = (X_0, Y, B_0)$ . When set different vertices to be principal coefficients seeds, it is conjectured that the  $\mathbf{g}$ -vectors with respect to a fixed vertex  $t$  of  $\mathbb{T}_n$  have the following relation.

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**Conjecture 1.3.** ([9], Conjecture 7.12) Let  $t_1 \dashrightarrow^k t_2 \in \mathbb{T}_n$  and let  $B^2 = \mu_k(B^1)$ . For  $a \in [1, n]$  and  $t \in \mathbb{T}_n$ , assume  $\mathbf{g}_{a;t}^{B^1;t_1} = (g_1^{t_1}, \dots, g_n^{t_1})$  and  $\mathbf{g}_{a;t}^{B^2;t_2} = (g_1^{t_2}, \dots, g_n^{t_2})$ , then

$$(1) \quad g_i^{t_2} = \begin{cases} -g_k^{t_1} & \text{if } i = k; \\ g_i^{t_1} + [b_{ik}^{t_1}] + g_k^{t_1} - b_{ik}^{t_1} \min(g_k^{t_1}, 0) & \text{if } i \neq k. \end{cases}$$

*Remark 1.4.* (1) As is said in Remark 7.14 of [9], it is easy to see that Conjectures 1.1 and 1.3 imply Conjecture 1.2.

(2) In the skew-symmetrizable case, the sign-coherence of the  $\mathbf{c}$ -vectors can deduce Conjectures 1.1 and 1.3, see [13], where the given method strongly depends on the skew-symmetrizability. So far, the similar conclusion have not been given in the sign-skew-symmetric case. Further, the sign-coherence of the  $\mathbf{c}$ -vectors has been proved in [12] for the acyclic sign-skew-symmetric case, because this is equivalent to  $F$ -polynomial has constant term 1, see [9]. Therefore, for the acyclic sign-skew-symmetric case, it is interesting to study directly Conjectures 1.1 and 1.3.

Unfolding of skew-symmetrizable matrices is introduced by Zelevinsky, whose aim is to characterize skew-symmetrizable cluster algebras using the version in skew-symmetric case. The second and third authors of this paper improved in [12] such method to arbitrary sign-skew-symmetric matrices. According to this previous work, there is an unfolding  $(\tilde{Q}, F, \Gamma)$  of any acyclic  $m \times n$  matrix  $\tilde{B}$ , and also a 2-Calabi-Yau Frobenius category  $\mathcal{C}^{\tilde{Q}}$  with  $\Gamma$  action constructed.

Our motivation and the main results of this paper are two-fold.

(1) Give the  $\Gamma$ -equivariant cluster character for  $\underline{\mathcal{C}^{\tilde{Q}}}$ , which can be regarded as the additive categorification of the cluster algebra  $\mathcal{A}(\Sigma(Q))$ . See Theorem 3.5 and Theorem 3.6.

(2) Solve Conjectures 1.1 and 1.3 in the acyclic case. See Theorem 4.7 and Theorem 5.5. As a consequence, in the same case, Conjecture 1.2 follows to be affirmed.

## 2. AN OVERVIEW OF UNFOLDING METHOD

In this section, we give a brief introduction of the concept of unfolding of totally sign-skew-symmetric cluster algebras and some necessary results in [12].

For any sign-skew-symmetric matrix  $B \in \text{Mat}_{n \times n}(\mathbb{Z})$ , one defines a quiver  $\Delta(B)$  as follows: the vertices are  $1, \dots, n$  and there is an arrow from  $i$  to  $j$  if and only if  $b_{ij} > 0$ .  $B$  is called **acyclic** if  $\Delta(B)$  is acyclic, a cluster algebra is called **acyclic** if it has an acyclic exchange matrix, see [3].

A locally finite **ice quiver** is a pair  $(Q, F)$  where  $Q$  is a locally finite quiver without 2-cycles or loops and  $F \subseteq Q_0$  is a subset of vertices called **frozen vertices** such that there are no arrows among vertices of  $F$ . For a locally finite ice quiver  $(Q, F)$ , we can associate an (infinite) skew-symmetric row and column finite (i.e. having at most finite nonzero entries in each row and column) matrix  $(b_{ij})_{i \in Q_0, j \in Q_0 \setminus F}$ , where  $b_{ij}$  equals to the number of arrows from  $i$  to  $j$  minus the number of arrows from  $j$  to  $i$ . In case of no confusion, for convenience, we also denote the ice quiver  $(Q, F)$  as  $(b_{ij})_{i \in Q_0, j \in Q_0 \setminus F}$ .

We say that an ice quiver  $(Q, F)$  admits the action of a group  $\Gamma$  if  $\Gamma$  acts on  $Q$  such that  $F$  is stable under the action. Let  $(Q, F)$  be a locally finite ice quiver with an action of a group  $\Gamma$  (maybe infinite). For a vertex  $i \in Q_0 \setminus F$ , a  $\Gamma$ -**loop** at  $i$  is an arrow from  $i$  to  $h \cdot i$  for some  $h \in \Gamma$ , a  $\Gamma$ -**2-cycle** at  $i$  is a pair of arrows  $i \rightarrow j$  and  $j \rightarrow h \cdot i$  for some  $j \notin \{h' \cdot i \mid h' \in \Gamma\}$  and  $h \in \Gamma$ . Denote by  $[i]$  the orbit set of  $i$  under the action of  $\Gamma$ . Say that  $(Q, F)$  has **no  $\Gamma$ -loops** (respectively,  **$\Gamma$ -2-cycles**) at  $[i]$  if  $(Q, F)$  has no  $\Gamma$ -loops ( $\Gamma$ -2-cycles, respectively) at any  $i' \in [i]$ .

**Definition 2.1.** (Definition 2.1, [12]) Let  $(Q, F) = (b_{ij})$  be a locally finite ice quiver with an group  $\Gamma$  action. Denote  $[i] = \{h \cdot i \mid h \in \Gamma\}$  the orbit of vertex  $i \in Q_0 \setminus F$ . Assume that  $(Q, F)$  admits no  $\Gamma$ -loops and no  $\Gamma$ -2-cycles at  $[i]$ , we define an **adjacent ice quiver**  $(Q', F) = (b'_{ij})_{i,j \in Q_0 \setminus F}$  from  $(Q, F)$  to be the quiver by following:

- (1) The vertices are the same as  $Q$ ,
- (2) The arrows are defined as

$$b'_{jk} = \begin{cases} -b_{jk}, & \text{if } j \in [i] \text{ or } k \in [i], \\ b_{jk} + \sum_{i' \in [i]} \frac{|b_{ji'}|b_{i'k} + b_{ji'}|b_{i'k}|}{2}, & \text{otherwise.} \end{cases}$$

Denote  $(Q', F)$  as  $\tilde{\mu}_{[i]}((Q, F))$  and call  $\tilde{\mu}_{[i]}$  the **orbit mutation** at direction  $[i]$  or at  $i$  under the action  $\Gamma$ . In this case, we say that  $(Q, F)$  can **do orbit mutation at  $[i]$** .

Note that if  $\Gamma$  is the trivial group  $\{e\}$ , then the definition of orbit mutation of a quiver is the same as that of quiver mutation (see [7][8]).

**Definition 2.2.** (Definition 2.4, [12]) (i) For a locally finite ice quiver  $(Q, F) = (b_{ij})_{i \in Q_0, j \in Q_0 \setminus F}$  with a group  $\Gamma$  (maybe infinite) action, let  $\overline{Q_0}$  (respectively,  $\overline{F}$ ) be the orbit sets of the vertex set  $Q_0$  (respectively, the frozen vertex set  $F$ ) under the  $\Gamma$ -action. Assume that  $m = |\overline{Q_0}| < +\infty$ ,  $m - n = |\overline{F}|$  and  $Q$  has no  $\Gamma$ -loops and no  $\Gamma$ -2-cycles.

Define a sign-skew-symmetric matrix  $B(Q, F) = (b_{[i][j]})_{[i] \in \overline{Q_0}, [j] \in \overline{Q_0} \setminus \overline{F}}$  to  $(Q, F)$  satisfying (1) the size of the matrix  $B(Q, F)$  is  $m \times n$ ; (2)  $b_{[i][j]} = \sum_{i' \in [i]} b_{i'j}$  for  $[i] \in \overline{Q_0}$ ,  $[j] \in \overline{Q_0} \setminus \overline{F}$ .

(ii) For an  $m \times n$  sign-skew-symmetric matrix  $B$ , if there is a locally finite ice quiver  $(Q, F)$  with a group  $\Gamma$  such that  $B = B(Q, F)$  as constructed in (i), then we call  $(Q, F, \Gamma)$  a **covering** of  $B$ .

(iii) For an  $m \times n$  sign-skew-symmetric matrix  $B$ , if there is a locally finite quiver  $(Q, F)$  with an action of group  $\Gamma$  such that  $(Q, F, \Gamma)$  is a covering of  $B$  and  $(Q, F)$  can do arbitrary steps of orbit mutations, then  $(Q, F, \Gamma)$  is called an **unfolding** of  $B$ ; or equivalently,  $B$  is called the **folding** of  $(Q, F, \Gamma)$ .

*Remark 2.3.* The definition of unfolding is slight different with that in [12] where the definition was just applied to square matrices.

By Lemma 2.5 of [12], we have the following consequence.

**Lemma 2.4.** *If  $(Q, F, \Gamma)$  is an unfolding of  $B$ , for any sequence  $[i_1], \dots, [i_s]$  of orbits of  $Q_0 \setminus F$  under the action of  $\Gamma$ , then  $(\tilde{\mu}_{[i_s]} \cdots \tilde{\mu}_{[i_1]}(Q, F), \Gamma)$  is a covering of  $\mu_{[i_s]} \cdots \mu_{[i_1]}B$ .*

By Theorem 2.16 of [12], we have

**Theorem 2.5.** *If  $\tilde{B} \in \text{Mat}_{m \times n}(\mathbb{Z})$  ( $m \geq n$ ) is an acyclic sign-skew-symmetric matrix, then  $\tilde{B}$  has an unfolding  $(\tilde{Q}, F, \Gamma)$ , where  $\tilde{Q}$  is given using of Construction 2.6 in [12].*

*Proof.* Assume  $\tilde{B} = \begin{pmatrix} B \\ B' \end{pmatrix}$  with  $B \in \text{Mat}_{n \times n}(\mathbb{Z})$ . Denote  $\tilde{B}' = \begin{pmatrix} B & -B'^T \\ B' & 0 \end{pmatrix}$ . Since  $\tilde{B}$  is acyclic,  $\tilde{B}' \in \text{Mat}_{m \times m}(\mathbb{Z})$  is acyclic. According to Construction 2.6 and Theorem 2.16 of [12],  $\tilde{B}'$  has an unfolding  $(\tilde{Q}, \Gamma)$ . Let  $F \subset \tilde{Q}_0$  be the vertices of  $\tilde{Q}_0$  corresponding to  $B'$ . Thus, it is clear that  $(\tilde{Q}, F, \Gamma)$  is an unfolding of  $\tilde{B}$ .  $\square$

*Remark 2.6.* In [12], we proved that  $\tilde{Q}$  is **strongly almost finite**, that is,  $\tilde{Q}$  is locally finite and has no path of infinite length.

This theorem means that an acyclic matrix is always totally sign-skew-symmetric. Thus, we can define a cluster algebra via an acyclic matrix.

For an acyclic matrix  $\tilde{B} \in \text{Mat}_{m \times n}(\mathbb{Z})$  ( $m \geq n$ ), assume  $(\tilde{Q}, F, \Gamma)$  is an unfolding of  $\tilde{B}$ . Denote  $\overline{\tilde{Q}_0}$  and  $\overline{F}$  be the orbits sets of vertices in  $\tilde{Q}_0$  and  $F$ . Let  $\tilde{\Sigma} = \Sigma(\tilde{Q}) = (\tilde{X}, \tilde{Y}, \tilde{Q})$  be the seed associated with  $(\tilde{Q}, F)$ , where  $\tilde{X} = \{x_u \mid u \in \tilde{Q}_0 \setminus F\}$ ,  $\tilde{Y} = \{y_v \mid v \in F\}$ . Let  $\Sigma = \Sigma(\tilde{B}) = (X, Y, \tilde{B})$  be the seed associated with  $\tilde{B}$ , where  $X = \{x_{[i]} \mid [i] \in \overline{\tilde{Q}_0} \setminus \overline{F}\}$ ,  $Y = \{y_{[j]} \mid [j] \in \overline{F}\}$ . It is clear that there is a surjective algebra homomorphism:

$$(2) \quad \pi : \mathbb{Q}[x_i^{\pm 1}, y_j \mid i \in \tilde{Q}_0 \setminus F, j \in F] \rightarrow \mathbb{Q}[x_{[i]}^{\pm 1}, y_{[j]} \mid [i] \in \overline{\tilde{Q}_0} \setminus \overline{F}, [j] \in \overline{F}]$$

such that  $\pi(x_i) = x_{[i]}$  and  $\pi(y_j) = y_{[j]}$ .

For any cluster variable  $x_u \in \tilde{X}$ , define  $\tilde{\mu}_{[i]}(x_u) = \mu_u(x_u)$  if  $u \in [i]$ ; otherwise,  $\tilde{\mu}_{[i]}(x_u) = x_u$  if  $u \notin [i]$ . Formally, write  $\tilde{\mu}_{[i]}(\tilde{X}) = \{\tilde{\mu}_{[i]}(x) \mid x \in \tilde{X}\}$  and  $\tilde{\mu}_{[i]}(\tilde{X}^{\pm 1}) = \{\tilde{\mu}_{[i]}(x)^{\pm 1} \mid x \in \tilde{X}\}$ .

**Lemma 2.7.** (Lemma 7.1, [12]) *Keep the notations as above. Assume that  $B$  is acyclic. If  $[i]$  is an orbit of vertices with  $i \in \tilde{Q}_0 \setminus F$ , then*

- (1)  $\tilde{\mu}_{[i]}(x_j)$  is a cluster variable of  $\mathcal{A}(\tilde{Q})$  for any  $j \in \tilde{Q}_0 \setminus F$ ,
- (2)  $\tilde{\mu}_{[i]}(\tilde{X})$  is algebraic independent over  $\mathbb{Q}[y_j \mid j \in F]$ .

By Lemma 2.7,  $\tilde{\mu}_{[i]}(\tilde{\Sigma}) := (\tilde{\mu}_{[i]}(\tilde{X}), \tilde{Y}, \tilde{\mu}_{[i]}(\tilde{Q}))$  is a seed. Thus, we can define  $\tilde{\mu}_{[i_s]} \tilde{\mu}_{[i_{s-1}]} \cdots \tilde{\mu}_{[i_1]}(x)$  and  $\tilde{\mu}_{[i_s]} \tilde{\mu}_{[i_{s-1}]} \cdots \tilde{\mu}_{[i_1]}(\tilde{X})$  and  $\tilde{\mu}_{[i_s]} \tilde{\mu}_{[i_{s-1}]} \cdots \tilde{\mu}_{[i_1]}(\tilde{\Sigma})$  for any sequence  $([i_1], [i_2], \dots, [i_s])$  of orbits in  $\tilde{Q}_0$ .

**Theorem 2.8.** (Theorem 7.5, [12]) *Keep the notations as above with an acyclic sign-skew-symmetric matrix  $B$  and  $\pi$  as defined in (2). Restricting  $\pi$  to  $\mathcal{A}(\tilde{\Sigma})$ , then  $\pi : \mathcal{A}(\tilde{\Sigma}) \rightarrow \mathcal{A}(\Sigma)$  is a surjective algebra morphism satisfying that  $\pi(\tilde{\mu}_{[j_k]} \cdots \tilde{\mu}_{[j_1]}(x_a)) = \mu_{[j_k]} \cdots \mu_{[j_1]}(x_{[a]}) \in \mathcal{A}(\Sigma)$  and  $\pi(\tilde{\mu}_{[j_k]} \cdots \tilde{\mu}_{[j_1]}(\tilde{X})) = \mu_{[j_k]} \cdots \mu_{[j_1]}(X)$  for any sequences of orbits  $[j_1], \dots, [j_k]$  and any  $a \in [i]$ .*

In case  $\mathcal{A}(\Sigma)$  with principal coefficients, from Lemma 2.4, we may assume that  $\mathcal{A}(\tilde{\Sigma})$  is also with principal coefficients. Let  $\lambda : \bigoplus_{i \in \tilde{Q}_0 \setminus F} \mathbb{Z}e_i \rightarrow \bigoplus_{[i] \in \overline{\tilde{Q}_0} \setminus \overline{F}} \mathbb{Z}e_{[i]}, e_i \rightarrow e_{[i]}$  be the group homomorphism, where  $\bigoplus_{i \in \tilde{Q}_0 \setminus F} \mathbb{Z}e_i$  (resp.  $\bigoplus_{[i] \in \overline{\tilde{Q}_0} \setminus \overline{F}} \mathbb{Z}e_{[i]}$ ) is the free abelian group generated by  $\{e_i \mid i \in \tilde{Q}_0 \setminus F\}$  (resp.  $\{e_{[i]} \mid [i] \in \overline{\tilde{Q}_0} \setminus \overline{F}\}$ ). Under such group homomorphism,  $\mathcal{A}(\tilde{\Sigma})$  becomes a  $\mathbb{Z}^n$ -graded algebra such any cluster variable  $x$  is homogenous with degree  $\lambda(\mathbf{g}(x))$ .

**Theorem 2.9.** *Keep the notations as in Theorem 2.8. If  $\mathcal{A}(\Sigma)$  with principal coefficients, then the restriction of  $\pi$  to  $\mathcal{A}(\tilde{\Sigma})$  is a  $\mathbb{Z}^n$ -graded surjective homomorphism.*

*Proof.* Since  $\lambda(\mathbf{g}(x_i)) = e_{[i]} = \mathbf{g}(x_{[i]})$  and  $\lambda(\mathbf{g}(y_j)) = -b_{[j]} = \mathbf{g}(y_{[j]})$ , where  $b_{[j]}$  is the  $[j]$ -th column of  $B$ . Further, because  $\{x_i^{\pm 1}, y_j \mid i \in \tilde{Q}_0 \setminus F, j \in \tilde{Q}_0\}$  is a generator of  $\mathbb{Q}[x_i^{\pm 1}, y_j \mid i \in \tilde{Q}_0 \setminus F, j \in \tilde{Q}_0]$ , thus  $\pi$  is homogenous. Then our result follows by Theorem 2.8.  $\square$

### 3. CLUSTER CHARACTER IN SIGN-SKEW-SYMMETRIC CASE

Let  $\tilde{B} \in \text{Mat}_{m \times n}(\mathbb{Z})$  be an acyclic sign-skew-symmetric matrix,  $(\tilde{Q}, F, \Gamma)$  be an unfolding of  $\tilde{B}$  given in Theorem 2.5. Denote  $\tilde{\Sigma} = (\tilde{X}, \tilde{Y}, \tilde{Q})$  and  $\Sigma = (X, Y, \tilde{B})$  be the seeds corresponding to  $(\tilde{Q}, F)$  and  $\tilde{B}$ .

From  $(\tilde{Q}, F, \Gamma)$ , we constructed a 2-Calabi-Yau Frobenius category  $\mathcal{C}^{\tilde{Q}}$  in [12] such that  $\Gamma$  acts on it exactly, i.e. each  $h \in \Gamma$  acts on  $\mathcal{C}^{\tilde{Q}}$  as an exact functor. Furthermore, there exists a cluster tilting subcategory  $\mathcal{T}_0$  of  $\mathcal{C}^{\tilde{Q}}$  such that the Gabriel quiver of  $\mathcal{T}_0$  is isomorphic to  $\tilde{Q}$ , where  $\mathcal{T}_0$  is the

subcategory of the stable category  $\underline{\mathcal{C}}^{\tilde{Q}}$  corresponding to  $\mathcal{T}_0$ . For details, see Lemma 4.15 of [12]. Since  $\mathcal{C}^{\tilde{Q}}$  is a *Hom*-finite 2-Calabi-Yau Frobenius category,  $\underline{\mathcal{C}}^{\tilde{Q}}$  follows to be a *Hom*-finite 2-Calabi-Yau triangulated category. Write  $[1]$  as the shift functor in  $\underline{\mathcal{C}}^{\tilde{Q}}$ . For any object  $X$  and subcategory  $\mathcal{X}$  of  $\underline{\mathcal{C}}^{\tilde{Q}}$ , we denote by  $\underline{X}$  and  $\underline{\mathcal{X}}$  the corresponding object and subcategory of  $\underline{\mathcal{C}}^{\tilde{Q}}$  respectively. Since the action of the group  $\Gamma$  on  $\mathcal{C}^{\tilde{Q}}$  is exact,  $\underline{\mathcal{C}}^{\tilde{Q}}$  also admits an exact  $\Gamma$ -action.

Since  $\mathcal{C}^{\tilde{Q}}$  is a Frobenius category, by the standard result, see [1], we have that

**Lemma 3.1.**  $Ext_{\underline{\mathcal{C}}^{\tilde{Q}}}^1(Z_1, Z_2) \cong Ext_{\underline{\mathcal{C}}^{\tilde{Q}}}^1(\underline{Z}_1, \underline{Z}_2)$  for all  $Z_1, Z_2 \in \mathcal{C}^{\tilde{Q}}$ .

The category  $\underline{\mathcal{C}}^{\tilde{Q}}$  can be viewed as an additive categorification of the cluster algebra  $\mathcal{A}(\tilde{\Sigma})$  given from  $\tilde{Q}$ . For details, refer to [10] and [15]. Although the authors of [10] and [15] deal with the cluster algebras of finite ranks, it is easy to see that these results still hold in  $\underline{\mathcal{C}}^{\tilde{Q}}$  since  $Q''$ , as well as  $\tilde{Q}$ , is a strongly almost finite quiver.

Denote  $\underline{\mathcal{T}}_0 = add(\underline{\mathcal{T}}' \cup \underline{\mathcal{T}}'')$ , where  $\underline{\mathcal{T}}'$  and  $\underline{\mathcal{T}}''$  respectively consist of the indecomposable objects correspondent to cluster variables in the clusters  $\tilde{X}$  and  $\tilde{Y}$  of  $\tilde{\Sigma}$ .

Let  $\mathcal{U}$  be the subcategory of  $\underline{\mathcal{C}}^{\tilde{Q}}$  generated by  $\{\underline{X} \in \underline{\mathcal{C}}^{\tilde{Q}} \mid Hom_{\underline{\mathcal{C}}^{\tilde{Q}}}(\underline{T}[-1], \underline{X}) = 0, \forall \underline{T} \in \underline{\mathcal{T}}''\}$ .

For any  $\underline{X} \in \mathcal{U}$ , let  $\underline{T}_1 \rightarrow \underline{T}_0 \xrightarrow{f} \underline{X} \rightarrow \underline{T}_1[1]$  be the triangle with  $f$  the minimal right  $\underline{\mathcal{T}}_0$ -approximation. By Lemma 3.1,  $\underline{\mathcal{T}}_0$  is a cluster tilting subcategory of  $\underline{\mathcal{C}}^{\tilde{Q}}$ . Applying  $Hom_{\underline{\mathcal{C}}^{\tilde{Q}}}(\underline{T}, -)$  to the triangle for all  $\underline{T} \in \underline{\mathcal{T}}_0$ , we have  $\underline{T}_1 \in \underline{\mathcal{T}}_0$ . The index of  $\underline{X}$  is defined as

$$ind_{\underline{\mathcal{T}}_0}(\underline{X}) = [\underline{T}_0] - [\underline{T}_1] \in K_0(\underline{\mathcal{T}}_0) \cong \mathbb{Z}^{|\tilde{Q}_0|}.$$

Recall that the Gabriel quiver of  $\underline{\mathcal{T}}_0$  is isomorphic to  $\tilde{Q}$ . We may assume that  $\{X_i \mid i \in \tilde{Q}_0\}$  is the complete set of the indecomposable objects of  $\underline{\mathcal{T}}_0$ . For any  $L \in mod \underline{\mathcal{T}}_0$ , we denote  $(dim_k(L(X_i)))_{i \in \tilde{Q}_0} \in \bigoplus_{i \in \tilde{Q}_0} \mathbb{Z}^{e_i}$  as its dimensional vector.

After the preparations, we give the definition of cluster character  $CC(\ )$  on  $\underline{\mathcal{C}}^{\tilde{Q}}$ . For any  $i \in \tilde{Q}_0 \setminus F$ , since  $\tilde{Q}$  is strongly almost finite, we can set

$$\hat{y}_i = \prod_{j \in \tilde{Q}_0 \setminus F} x_j^{b_{ji}} \prod_{j' \in F} y_{j'}^{b_{j'i}} \in \mathbb{Q}[x_i^{\pm 1}, y_j \mid i \in \tilde{Q}_0 \setminus F, j \in F].$$

For each rigid object  $\underline{X} \in \mathcal{U}$ , we define

$$CC(\underline{X}) = \tilde{\mathbf{x}}^{ind_{\underline{\mathcal{T}}_0}(\underline{X})} \sum_{\substack{\mathbf{a} \in \bigoplus_{i \in \tilde{Q}_0} \mathbb{Z} e_i}} \chi(Gr_{\mathbf{a}}(Hom_{\underline{\mathcal{C}}^{\tilde{Q}}}(-, \underline{X}[1]))) \prod_{j \in \tilde{Q}_0 \setminus F} \hat{y}_j^{a_j},$$

where  $\tilde{\mathbf{x}}^{\mathbf{a}} = \prod x_i^{a_i}$  for  $\mathbf{a} = (a_i)_{i \in \tilde{Q}_0} \in \bigoplus_{i \in \tilde{Q}_0} \mathbb{Z} e_i$ ,  $Gr_{\mathbf{a}}(Hom_{\underline{\mathcal{C}}^{\tilde{Q}}}(-, \underline{X}[1]))$  is the quiver Grassmannian whose points are corresponding to the sub- $\underline{\mathcal{T}}_0$ -representations of  $Hom_{\underline{\mathcal{C}}^{\tilde{Q}}}(-, \underline{X}[1])$  with dimension vector  $\mathbf{a}$ , and  $\chi$  is the Euler characteristic with respect to étale cohomology with proper support. It is easy to see that  $CC(\underline{X}) \in \mathbb{Q}[x_i^{\pm 1}, y_j \mid i \in \tilde{Q}_0 \setminus F, j \in F]$ .

**Theorem 3.2.** *Keeps the notations as above. Then*

- (1)  $CC(\underline{T}_i) = x_i$  for all  $\underline{T}_i \in \underline{\mathcal{T}}'$ .
- (2)  $CC(\underline{X} \oplus \underline{X}') = CC(\underline{X})CC(\underline{X}')$  for any objects  $\underline{X}, \underline{X}' \in \mathcal{U}$ .
- (3)  $CC(\underline{X})CC(\underline{Y}) = CC(\underline{Z}) + CC(\underline{Z}')$  for  $\underline{X}, \underline{Y} \in \mathcal{U}$  with  $dim Ext_{\underline{\mathcal{C}}^{\tilde{Q}}}^1(\underline{X}, \underline{Y}) = 1$  and the two non-splitting triangles:  $\underline{Y} \rightarrow \underline{Z} \rightarrow \underline{X} \rightarrow \underline{Y}[1]$  and  $\underline{X} \rightarrow \underline{Z}' \rightarrow \underline{Y} \rightarrow \underline{X}[1]$ .

*Proof.* This theorem can be proved similarly as that of ([10], Theorem 3.3) using of local finiteness of  $\tilde{Q}$  since it is strongly almost finite.  $\square$

Like [5], for any  $\underline{X} \in \mathcal{U}$ , we define  $ind'_{\underline{T}_0}(\underline{X}) = \lambda(ind_{\underline{T}_0}(\underline{X})) \in \bigoplus_{[i] \in \overline{Q_0}} \mathbb{Z}e_{[i]}$ .

For each rigid object  $\underline{X} \in \mathcal{U}$ , using the definition of  $\pi$  in (2), we define the  $\Gamma$ -equivariant cluster character as follows:

$$CC'(\underline{X}) = \pi(CC(\underline{X})) = \mathbf{x}^{ind'_{\underline{T}_0} \underline{X}} \sum_{\mathbf{a} \in \bigoplus_{i \in \overline{Q_0}} \mathbb{Z}e_i} \chi(Gr_{\mathbf{a}} Hom_{\underline{\mathcal{C}\tilde{Q}}}(-, \underline{X}[1])) \prod_{[j] \in \overline{Q_0} \setminus \overline{F}} \hat{y}_{[j]}^{\lambda(\mathbf{a})_{[j]}},$$

where  $\mathbf{x}^{\mathbf{a}} = \prod x_{[i]}^{a_{[i]}}$  for  $\mathbf{a} = (a_{[i]})_{[i] \in \overline{Q_0} \setminus \overline{F}} \in \bigoplus_{[i] \in \overline{Q_0} \setminus \overline{F}} \mathbb{Z}e_{[i]}$  and  $\hat{y}_{[i]} = \pi(\hat{y}_i) = \prod_{[j] \in \overline{Q_0} \setminus \overline{F}} x_{[j]}^{b_{[j]i}} \prod_{[j'] \in \overline{Q_0} \setminus \overline{F}} y_{[j']}^{b_{[j']i}}$ .

Inspired by Definition 3.34 of [5], we give the following definition,

**Definition 3.3.** Two objects  $\underline{X}, \underline{X}' \in \underline{\mathcal{C}\tilde{Q}}$  are said to be **equivalent modulo  $\Gamma$**  if  $\underline{X} \cong \bigoplus_{k=1}^m \underline{X}_k$ ,  $\underline{X}' \cong \bigoplus_{k=1}^m \underline{X}'_k$  with  $add\{h \cdot \underline{X}_k \mid h \in \Gamma\} = add\{h \cdot \underline{X}'_k \mid h \in \Gamma\}$  for every  $k$  and indecomposables  $\underline{X}_k, \underline{X}'_k$ .

Similar to Lemma 3.49 of [5], we have the following lemma,

**Lemma 3.4.** *The Laurent polynomial  $CC'(\underline{X})$  depends only on the class of  $\underline{X}$  under equivalent modulo  $\Gamma$  for and  $\underline{X} \in \underline{\mathcal{C}\tilde{Q}}$ .*

*Proof.* We need only to prove that  $CC'(\underline{X}) = CC'(h \cdot \underline{X})$  for  $h \in \Gamma$  by Theorem 3.2(2). It follows immediately from that  $ind'_{\underline{T}_0}(\underline{X}) = ind'_{\underline{T}_0}(h \cdot \underline{X})$  and  $\chi(Gr_{\mathbf{a}} Hom_{\underline{\mathcal{C}\tilde{Q}}}(-, \underline{X}[1])) = \chi(Gr_{h \cdot \mathbf{a}} Hom_{\underline{\mathcal{C}\tilde{Q}}}(-, h \cdot \underline{X}[1]))$ .  $\square$

Using the algebra homomorphism  $\pi$  and Theorem 3.2, we have the following theorem at once,

**Theorem 3.5.** *Keeps the notations as above. Then*

- (1)  $CC'(\underline{T}_i) = x_{[i]}$  for all  $\underline{T}_i \in \underline{\mathcal{T}}$ .
- (2)  $CC'(\underline{X} \oplus \underline{X}') = CC'(\underline{X})CC'(\underline{X}')$  for any two objects  $\underline{X}, \underline{X}' \in \mathcal{U}$ .
- (3)  $CC'(\underline{X})CC'(\underline{Y}) = CC'(\underline{Z}) + CC'(\underline{Z}')$  for  $\underline{X}, \underline{Y} \in \mathcal{U}$  with  $dim Ext_{\underline{\mathcal{C}\tilde{Q}}}^1(\underline{X}, \underline{Y}) = 1$  satisfying two non-splitting triangles:  $\underline{Y} \rightarrow \underline{Z} \rightarrow \underline{X} \rightarrow \underline{Y}[1]$  and  $\underline{X} \rightarrow \underline{Z}' \rightarrow \underline{Y} \rightarrow \underline{X}[1]$ .

Following [14], we say  $\underline{X} \in \underline{\mathcal{C}\tilde{Q}}$  to be **reachable** if it belongs to a cluster-tilting subcategory which can be obtained by a sequence of mutations from  $\underline{\mathcal{T}_0}$  with the mutations do not take at the objects in  $\underline{\mathcal{T}''}$ . It is clear that any reachable object belongs to  $\mathcal{U}$ .

Following Theorem 4.1 of [14], we have the following result:

**Theorem 3.6.** *Keep the notations as above. Then the cluster character  $CC'(\ )$  gives a surjection from the set of equivalence classes of indecomposable reachable objects under equivalent modulo  $\Gamma$  of  $\underline{\mathcal{C}\tilde{Q}}$  to the set of clusters variables of the cluster algebra  $\mathcal{A}(\Sigma)$ .*

*Proof.* By Theorem 3.5, the proof is similar as that of Theorem 4.1 in [14].  $\square$

#### 4. SIGN-COHERENCE OF $\mathbf{g}$ -VECTORS

Keep the notations in Section 3. In this section, we will prove the sign-coherence of  $\mathbf{g}$ -vectors for acyclic sign-skew-symmetric cluster algebras. For convenience, suppose that  $\mathcal{A}(\Sigma)$  is an acyclic sign-skew-symmetric cluster algebra with principal coefficients at  $\Sigma$ .

Since  $B$  is acyclic,  $\begin{pmatrix} B \\ I_n \end{pmatrix}$  is acyclic, too. We can construct an unfolding  $(\tilde{Q}, F, \Gamma)$  according to Theorem 2.5. It is easy to check that the corresponding seed  $\tilde{\Sigma}$  of  $(\tilde{Q}, F, \Gamma)$  is with principal coefficients, where  $\tilde{\Sigma}$  is the seed associate to  $(Q, F)$ .

**Proposition 4.1.** *Assume that  $\underline{X}$  is an object of  $\mathcal{U}$  given in Section 3.*

- (i)  $CC(\underline{X})$  admits a  $\mathbf{g}$ -vector  $(g_i)_{i \in \tilde{Q}_0 \setminus F}$  which is given by  $g_i = [\text{ind}_{\mathcal{T}_0}(\underline{X}) : \underline{T}_i]$  for each  $i$ .
- (ii)  $CC'(\underline{X})$  admits a  $\mathbf{g}$ -vector  $(g_{[i]})_{[i] \in \overline{Q_0} \setminus \overline{F}}$  which is given by  $g_{[i]} = \sum_{i' \in [i]} [\text{ind}_{\mathcal{T}_0}(\underline{X}) : \underline{T}_{i'}]$  for each  $i$ .

*Proof.* (i) Since the quiver  $\tilde{Q}$  is strongly almost finite, the proof is the same one as that of ([15], Proposition 3.6) using of the local finiteness of  $\tilde{Q}$ .

(ii) is obtained immediately from (i).  $\square$

Let  $h \in \Gamma$  be either of finite order or without fixed points. Define  $\underline{\mathcal{C}}_{\tilde{Q}_h}^{\tilde{Q}}$  the  $K$ -linear category whose objects are the same as that of  $\underline{\mathcal{C}}^{\tilde{Q}}$ , and whose morphisms consist of  $\text{Hom}_{\underline{\mathcal{C}}_{\tilde{Q}_h}^{\tilde{Q}}}(\underline{X}, \underline{Y}) = \bigoplus_{h' \in \Gamma'} \text{Hom}_{\underline{\mathcal{C}}^{\tilde{Q}}}(h' \cdot \underline{X}, \underline{Y})$  for all objects  $\underline{X}, \underline{Y}$ . We view this category as a dual construction of the category  $\mathcal{C}_h^Q$  in [12].

Denote by  $\underline{\mathcal{C}}_{\tilde{Q}_h}^{\tilde{Q}}(\underline{\mathcal{T}}_0)$  the subcategory of  $\underline{\mathcal{C}}_{\tilde{Q}_h}^{\tilde{Q}}$  consisting of all objects  $\underline{T} \in \underline{\mathcal{T}}_0$ .

**Lemma 4.2.** *The category  $\underline{\mathcal{C}}_{\tilde{Q}_h}^{\tilde{Q}}$  is Hom-finite if either (i) the order of  $h$  is finite, or (ii)  $Q$  has no fixed points under the action of  $h \in \Gamma$ .*

*Proof.* The proof is similar to that of Lemma 6.1 in [12].  $\square$

In the sequel, assume that  $\underline{\mathcal{C}}_{\tilde{Q}_h}^{\tilde{Q}}(\underline{\mathcal{T}}_0)$  is Hom-finite. Let  $F : \underline{\mathcal{C}}^{\tilde{Q}} \rightarrow \text{mod} \underline{\mathcal{C}}_{\tilde{Q}_h}^{\tilde{Q}}(\underline{\mathcal{T}}_0)^{op}$  be the functor mapping an object  $X$  to the restricting of  $\underline{\mathcal{C}}_{\tilde{Q}_h}^{\tilde{Q}}(-, X)$  to  $\underline{\mathcal{C}}_{\tilde{Q}_h}^{\tilde{Q}}(\underline{\mathcal{T}}_0)$ . For each indecomposable object  $\underline{T}$  of  $\underline{\mathcal{T}}_0$ , denote by  $S_{\underline{T}}$  the **simple quotient** of  $F(\underline{T})$  via  $S_{\underline{T}}(\underline{T}') = \text{End}_{\mathcal{C}}(\underline{T})/J$  if  $\underline{T}' \cong \underline{T}$  and  $S_{\underline{T}}(\underline{T}') = 0$  if  $\underline{T}' \not\cong \underline{T}$ , for any object  $\underline{T}'$  of  $\underline{\mathcal{C}}_{\tilde{Q}_h}^{\tilde{Q}}$ , where  $J$  is the Jacobson radical of  $\text{End}_{\underline{\mathcal{C}}_{\tilde{Q}_h}^{\tilde{Q}}}(\underline{T})$ .

**Lemma 4.3.** *Keep the notations as above. For any morphism  $\tilde{f} : F(\underline{M}) \rightarrow F(\underline{N})$  in  $\text{mod} \underline{\mathcal{C}}_{\tilde{Q}_h}^{\tilde{Q}}(\underline{\mathcal{T}}_0)^{op}$  with  $\underline{M}, \underline{N} \in \underline{\mathcal{C}}^{\tilde{Q}}$ , there exists  $f : \underline{M} \rightarrow \underline{N}$  in  $\underline{\mathcal{C}}_{\tilde{Q}_h}^{\tilde{Q}}$  such that  $F(f) = \tilde{f}$ .*

*Proof.* Let  $\underline{T}_1 \xrightarrow{e} \underline{T}_0 \xrightarrow{d} \underline{M} \rightarrow \underline{T}_1[1]$  be the triangle such that  $d$  is a minimal right  $\underline{\mathcal{T}}_0$ -approximation. Since  $\underline{\mathcal{T}}_0$  is a cluster tilting subcategory of  $\underline{\mathcal{C}}^{\tilde{Q}}$ , we have  $\underline{T}_1 \in \underline{\mathcal{T}}_0$ . Applying  $F$  to the above triangle, we have  $F(\underline{T}_1) \rightarrow F(\underline{T}_0) \rightarrow F(\underline{M}) \rightarrow 0$ . Similarly, there is a triangle  $\underline{T}'_1 \xrightarrow{e'} \underline{T}'_0 \xrightarrow{d'} \underline{N} \rightarrow \underline{T}'_1[1]$ , and  $F(\underline{T}'_1) \rightarrow F(\underline{T}'_0) \rightarrow F(\underline{N}) \rightarrow 0$ . Since  $\underline{T}_0, \underline{T}'_0, \underline{T}_1, \underline{T}'_1 \in \underline{\mathcal{T}}_0$ , it follows that  $F(\underline{T}_0), F(\underline{T}'_0), F(\underline{T}_1), F(\underline{T}'_1)$  are projective in  $\text{mod} \underline{\mathcal{C}}_{\tilde{Q}_h}^{\tilde{Q}}(\underline{\mathcal{T}}_0)^{op}$ . Thus,  $\tilde{f}$  can be lift to the following commutative diagram:

$$\begin{array}{ccccccc} F(\underline{T}_1) & \longrightarrow & F(\underline{T}_0) & \longrightarrow & F(\underline{M}) & \longrightarrow & 0 \\ \downarrow \tilde{f}_1 & & \downarrow \tilde{f}_0 & & \downarrow \tilde{f} & & \\ F(\underline{T}'_1) & \longrightarrow & F(\underline{T}'_0) & \longrightarrow & F(\underline{N}) & \longrightarrow & 0, \end{array}$$

By the Yoneda Lemma, there exist  $f_1 : \underline{T}_1 \rightarrow \underline{T}'_1$  and  $f_0 : \underline{T}_0 \rightarrow \underline{T}'_0$  in  $\underline{\mathcal{C}}_{\tilde{Q}_h}^{\tilde{Q}}$  such that  $F(f_1) = \tilde{f}_1$  and  $F(f_0) = \tilde{f}_0$ . Thus,  $f_1 \in \bigoplus_{h' \in \Gamma'} \text{Hom}_{\underline{\mathcal{C}}^{\tilde{Q}}}(h' \cdot \underline{T}_1, \underline{T}'_1)$  and  $f_0 \in \bigoplus_{h' \in \Gamma'} \text{Hom}_{\underline{\mathcal{C}}^{\tilde{Q}}}(h' \cdot \underline{T}_0, \underline{T}'_0)$  such that

$$\begin{array}{ccccccc} \bigoplus_{h' \in \Gamma'} h' \cdot \underline{T}_1 & \longrightarrow & \bigoplus_{h' \in \Gamma'} h' \cdot \underline{T}_0 & \longrightarrow & \bigoplus_{h' \in \Gamma'} h' \cdot \underline{M} & \longrightarrow & 0 \\ \downarrow f_1 & & \downarrow f_0 & & \downarrow & & \\ \underline{T}'_1 & \longrightarrow & \underline{T}'_0 & \longrightarrow & \underline{N} & \longrightarrow & 0, \end{array}$$

commutes. Since  $\underline{\mathcal{C}}_{\tilde{Q}_h}(\underline{\mathcal{T}}_0)$  is *Hom*-finite, we may assume that only finite  $h' \in \Gamma'$  appear in the upper triangle, which means that there exists a finite subset  $I$  of  $\Gamma'$  such that

$$\begin{array}{ccccccc} \bigoplus_{h' \in I} h' \cdot \underline{\mathcal{T}}_1 & \longrightarrow & \bigoplus_{h' \in I} h' \cdot \underline{\mathcal{T}}_0 & \longrightarrow & \bigoplus_{h' \in I} h' \cdot \underline{M} & \longrightarrow & 0 \\ \downarrow f_1 & & \downarrow f_0 & & \downarrow & & \\ \underline{\mathcal{T}}'_1 & \longrightarrow & \underline{\mathcal{T}}'_0 & \longrightarrow & \underline{N} & \longrightarrow & 0, \end{array}$$

commutes. Thus, there exists  $f \in \bigoplus_{h' \in I} \text{Hom}_{\underline{\mathcal{C}}_{\tilde{Q}}}(\underline{h'} \cdot \underline{\mathcal{T}}_0, \underline{\mathcal{T}}'_0) \subseteq \bigoplus_{h' \in \Gamma'} \text{Hom}_{\underline{\mathcal{C}}_{\tilde{Q}}}(\underline{h'} \cdot \underline{\mathcal{T}}_0, \underline{\mathcal{T}}'_0)$  such that the above diagram commutes. This commutative diagram induces

$$\begin{array}{ccccccc} F(\underline{\mathcal{T}}_1) & \longrightarrow & F(\underline{\mathcal{T}}_0) & \longrightarrow & F(\underline{M}) & \longrightarrow & 0 \\ \downarrow \tilde{f}_1 & & \downarrow \tilde{f}_0 & & \downarrow F(f) & & \\ F(\underline{\mathcal{T}}'_1) & \longrightarrow & F(\underline{\mathcal{T}}'_0) & \longrightarrow & F(\underline{N}) & \longrightarrow & 0. \end{array}$$

Therefore, we have  $F(f) = \tilde{f}$ . □

**Lemma 4.4.** *Assume that for  $\underline{X} \in \mathcal{U}$ , the category  $\text{add}(\{h \cdot \underline{X} \mid h \in \Gamma\})$  is rigid. Let  $\underline{\mathcal{T}}_1 \xrightarrow{f'} \underline{\mathcal{T}}_0 \xrightarrow{f} \underline{X} \rightarrow \underline{\mathcal{T}}_1[1]$  be a triangle in  $\underline{\mathcal{C}}_{\tilde{Q}}$  with  $f$  a minimal right  $\underline{\mathcal{T}}_0$ -approximation. If  $\underline{X}$  has not any direct summand in  $\underline{\mathcal{T}}_0[1]$ , then*

$$F(\underline{\mathcal{T}}_1) \xrightarrow{F(f')} F(\underline{\mathcal{T}}_0) \xrightarrow{F(f)} F(\underline{X}) \rightarrow 0$$

*is a minimal projective resolution of  $F(\underline{X})$ .*

*Proof.* Since  $\underline{X}$  does not have direct summand in  $\underline{\mathcal{T}}_0[1]$ ,  $f'$  is right minimal. Otherwise, if  $f'$  is not right minimal, then  $f'$  has a direct summand as  $\underline{\mathcal{T}}' \rightarrow 0$ , thus  $\underline{\mathcal{T}}'[1]$  is a direct summand of  $\underline{X}$ . This is a contradiction.

First, we prove that  $F(f)$  is a projective cover of  $F(\underline{X})$ . For any projective representation  $F(\underline{\mathcal{T}})e$  of  $\underline{\mathcal{C}}_{\tilde{Q}_h}$  and surjective morphism  $u : F(\underline{\mathcal{T}})e \rightarrow F(\underline{X}) \rightarrow 0$ , where  $\underline{\mathcal{T}} \in \underline{\mathcal{C}}_{\tilde{Q}_h}$ ,  $e \in \text{End}_{\underline{\mathcal{C}}_{\tilde{Q}_h}}(\underline{\mathcal{T}})$  is an idempotent. By Yoneda Lemma, there exists  $g \in \text{Hom}_{\underline{\mathcal{C}}_{\tilde{Q}_h}}(\underline{\mathcal{T}}, \underline{X})$  such that  $F(g) = u$ . Since  $F(\underline{\mathcal{T}})e$  and  $F(\underline{\mathcal{T}}_0)$  are projective, there exist  $v : F(\underline{\mathcal{T}}_0) \rightarrow F(\underline{\mathcal{T}})e$  and  $w : F(\underline{\mathcal{T}}) \rightarrow F(\underline{\mathcal{T}}_0)$  such that  $F(f) = F(g)v$  and  $F(g) = F(f)w$ . Thus,  $F(f) = F(f)wv$ . Similarly, by Yoneda Lemma, there exist  $g' \in \text{Hom}_{\underline{\mathcal{C}}_{\tilde{Q}_h}}(\underline{\mathcal{T}}_0, \underline{\mathcal{T}})$  and  $g'' \in \text{Hom}_{\underline{\mathcal{C}}_{\tilde{Q}_h}}(\underline{\mathcal{T}}, \underline{\mathcal{T}}_0)$  such that  $F(g') = v$ ,  $F(g'') = w$ . Thus,  $F(f) = F(f)wv$  is equivalent to  $f = f \circ g'' \circ g'$ . We may assume  $g'' \circ g' = (g_{h'})_{h' \in \Gamma'}$ , then  $f = f \circ (g'' \circ g') = (fg_{h'})_{h' \in \Gamma'}$ . Thus  $f = fg_e$  and  $0 = fg_{h'} = 0$  for any  $h' \neq e$ , where  $e$  is the identity of  $\Gamma'$ . Further, since  $f$  is a right minimal  $\underline{\mathcal{T}}_0$ -approximation and  $h'\underline{\mathcal{T}}_0 \in \underline{\mathcal{T}}_0$ . Therefore, for any  $e \neq h' \in \Gamma$ ,  $g_{h'} \in J(h'\underline{\mathcal{T}}_0, \underline{\mathcal{T}}_0)$  and  $g_e$  is an isomorphism. Using Lemma 5.10 of [12],  $g'' \circ g' = (g_{h'})_{h' \in \Gamma'}$  is an isomorphism. Thus,  $F(\underline{\mathcal{T}}_0)$  is a direct summand of  $F(\underline{\mathcal{T}})e$ .

Similarly, because  $f'$  is right minimal,  $F(f')$  induces a projective cover of  $\ker(F(f))$ . Our result follows. □

Inspired by (Proposition 2.1, [4]), (Lemma 3.58, [5]) and (Lemma 3.5, [15]), we have:

**Lemma 4.5.** *Assume  $\underline{X} \in \mathcal{U}$  such that  $\text{add}(\{h \cdot \underline{X} \mid h \in \Gamma\})$  is rigid, and  $\underline{\mathcal{T}}_1 \xrightarrow{f'} \underline{\mathcal{T}}_0 \xrightarrow{f} \underline{X} \rightarrow \underline{\mathcal{T}}_1[1]$  is a triangle in  $\underline{\mathcal{C}}_{\tilde{Q}}$  and  $f$  is a minimal right  $\underline{\mathcal{T}}_0$ -approximation. If  $\underline{\mathcal{T}}$  is a direct summand of  $\underline{\mathcal{T}}_0$  for indecomposable object  $\underline{\mathcal{T}} \in \underline{\mathcal{C}}_{\tilde{Q}}$ , then  $h \cdot \underline{\mathcal{T}}$  is not a direct summand of  $\underline{\mathcal{T}}_1$  for any  $h \in \Gamma$ .*

*Proof.* By Lemma 2.3 and Lemma 2.4 of [12], we may assume that  $h$  has no fixed points or finite order. By Lemma 4.2,  $\underline{\mathcal{C}}_{\tilde{Q}_h}$  is *Hom*-finite. For any  $h' \in \Gamma'$  and  $\underline{\mathcal{T}}' \in \underline{\mathcal{T}}$ , applying  $\text{Hom}_{\underline{\mathcal{C}}_{\tilde{Q}}}(\underline{h'} \cdot \underline{\mathcal{T}}', -)$



to the triangle, we get

$$\mathrm{Hom}_{\underline{\mathcal{C}}\tilde{\mathcal{Q}}}(h' \cdot \underline{T}', \underline{T}_1) \rightarrow \mathrm{Hom}_{\underline{\mathcal{C}}\tilde{\mathcal{Q}}}(h' \cdot \underline{T}', \underline{T}_0) \xrightarrow{\mathrm{Hom}_{\underline{\mathcal{C}}\tilde{\mathcal{Q}}}(h' \cdot \underline{T}', f)} \mathrm{Hom}_{\underline{\mathcal{C}}\tilde{\mathcal{Q}}}(h' \cdot \underline{T}', \underline{X}) \rightarrow 0.$$

Since  $f$  is minimal, we get a minimal projective resolution of  $F(\underline{X})$ ,

$$F(\underline{T}_1) \rightarrow F(\underline{T}_0) \rightarrow F(\underline{X}) \rightarrow 0.$$

To prove that  $h \cdot \underline{T}$  is not a direct summand of  $\underline{T}_1$ , it suffices to prove that  $F(\underline{T})$  is not a direct summand of  $F(\underline{T}_1)$ , or equivalently  $\mathrm{Ext}^1(F(\underline{X}), S_{\underline{T}}) = 0$ , where  $S_{\underline{T}}$  is the simple quotient of  $F(\underline{T})$ .

As  $F(\underline{T}_0) \rightarrow F(\underline{X})$  is the projective cover of  $F(\underline{X})$  and  $F(\underline{T})$  is a direct summand of  $F(\underline{T}_0)$ , then there is a non-zero morphism  $\tilde{p} : F(\underline{X}) \rightarrow S_{\underline{T}}$ . For any  $\tilde{g} : F(\underline{T}_1) \rightarrow S_{\underline{T}}$ , since  $F(\underline{T}_1)$  is projective, there exists  $\tilde{q} : F(\underline{T}_1) \rightarrow F(\underline{X})$  such that  $\tilde{g} = \tilde{p}\tilde{q}$ .

Let  $T$  be a lifting of  $\underline{T}$ , by Lemma 3.1, we have  $T \in \mathcal{T}$  and  $T$  is non-projective. Moreover, since  $\underline{T}$  is indecomposable, we can choose  $T$  is indecomposable. By Lemma 5.3 of [12], there is an admissible short exact sequence  $0 \rightarrow Y \rightarrow T' \rightarrow T \rightarrow 0$ . Since  $\mathcal{T}$  has no  $\Gamma$ -loop, by the dual version of Lemma 5.11 (2) of [12], and Lemma 3.1, we have  $S_{\underline{T}} \cong F(\underline{Y}[1])$ .

Thus, according to Lemma 4.3, lifting  $\tilde{q}$ ,  $\tilde{g}$  and  $\tilde{p}$  as  $q, g, p$  in  $\underline{\mathcal{C}}\tilde{\mathcal{Q}}_h$ , where  $q \in \bigoplus_{h' \in \Gamma'} \mathrm{Hom}_{\underline{\mathcal{C}}\tilde{\mathcal{Q}}}(h' \cdot \underline{T}_1, \underline{X})$ ,  $g \in \bigoplus_{h' \in \Gamma'} \mathrm{Hom}_{\underline{\mathcal{C}}\tilde{\mathcal{Q}}}(h' \cdot \underline{T}_1, \underline{Y}[1])$  and  $p \in \bigoplus_{h' \in \Gamma'} \mathrm{Hom}_{\underline{\mathcal{C}}\tilde{\mathcal{Q}}}(h' \cdot \underline{X}, \underline{Y}[1])$ . Since  $\tilde{g} = \tilde{p}\tilde{q}$  and  $\mathrm{Hom}(F(\underline{T}), F(\underline{Y}[1])) \cong \mathrm{Hom}_{\underline{\mathcal{C}}\tilde{\mathcal{Q}}_h}(\underline{T}, \underline{Y}[1])$ , we obtain  $g = p \circ q$ .

Since  $\underline{\mathcal{C}}\tilde{\mathcal{Q}}_h$  is  $\mathrm{Hom}$ -finite, we may assume that  $g \in \bigoplus_{h' \in I} \mathrm{Hom}_{\underline{\mathcal{C}}\tilde{\mathcal{Q}}}(h' \cdot \underline{T}_1, \underline{Y}[1])$  and  $q \in \bigoplus_{h' \in I} \mathrm{Hom}_{\underline{\mathcal{C}}\tilde{\mathcal{Q}}}(h' \cdot \underline{T}_1, \underline{X})$  for a finite subset  $I \subseteq \Gamma'$ . According to the composition of morphisms in  $\underline{\mathcal{C}}\tilde{\mathcal{Q}}_h$ ,  $g = p \circ q$  means  $g = p(\sum_{h' \in I} h' \cdot q)$ , equivalently, we have the following commutative diagram:

$$\begin{array}{ccccccc} \bigoplus_{h' \in I} h' \cdot \underline{X}[1] & \xrightarrow{\sum h' \cdot f'} & \bigoplus_{h' \in I} h' \cdot \underline{T}_1 & \xrightarrow{\sum h' \cdot f} & \bigoplus_{h' \in I} h' \cdot \underline{T}_0 & \longrightarrow & \bigoplus_{h' \in I} h' \cdot \underline{X} \\ & \searrow \sum h' \cdot q & \downarrow g & & & & \\ \bigoplus_{h' \in I} h' \cdot \underline{X} & \xrightarrow{p} & \underline{Y}[1], & & & & \end{array}$$

Since  $\mathrm{add}(\{h \cdot \underline{X} \mid h \in \Gamma\})$  is rigid, we have  $g(\sum h' \cdot f') = 0$ . Therefore,  $g$  factor through  $\sum h' \cdot f$ . Thus, in  $\underline{\mathcal{C}}\tilde{\mathcal{Q}}_h$ ,  $g$  factors through  $f$ . By the arbitrary of  $g$ , we get a surjective map  $\mathrm{Hom}_{\underline{\mathcal{C}}\tilde{\mathcal{Q}}_h}(\underline{T}_0, \underline{Y}[1]) \rightarrow \mathrm{Hom}_{\underline{\mathcal{C}}\tilde{\mathcal{Q}}_h}(\underline{T}_1, \underline{Y}[1])$ . Since  $\mathrm{Hom}_{\underline{\mathcal{C}}\tilde{\mathcal{Q}}_h}(\underline{T}_i, \underline{Y}[1]) \cong \mathrm{Hom}(F(\underline{T}_i), F(\underline{Y}[1])) = \mathrm{Hom}(F(\underline{T}_i), S_{\underline{T}})$  for  $i = 1, 2$ , we get  $\mathrm{Hom}(F(\underline{T}_0), S_{\underline{T}}) \rightarrow \mathrm{Hom}(F(\underline{T}_1), S_{\underline{T}})$  is surjective. Therefore, we obtain  $\mathrm{Ext}^1(F(\underline{X}), S_{\underline{T}}) = 0$ . Our result follows.  $\square$

**Corollary 4.6.** *Let  $\underline{X}$  be an object of  $\underline{\mathcal{C}}$  such that  $\mathrm{add}(\{h \cdot \underline{X} \mid h \in \Gamma\})$  is rigid. If  $\mathrm{ind}'_{\underline{T}_0}(\underline{X})$  has no negative coordinates, then  $\underline{X} \in \underline{\mathcal{T}}$ .*

*Proof.* Assume  $\underline{T}_1 \xrightarrow{f'} \underline{T}_0 \xrightarrow{f} \underline{X} \rightarrow \underline{T}_1[1]$  is a triangle and  $f$  is a minimal right  $\mathcal{T}$ -approximation. According to Lemma 4.5,  $\underline{T}_0$  and  $\underline{T}_1$  have no direct summands which have the same  $\Gamma$ -orbits. Further, since  $\mathrm{ind}'_{\underline{T}}(\underline{X})$  has no negative components. Therefore, by the definition of  $\mathrm{ind}'_{\underline{T}_0}$ , we obtain  $\underline{T}_1 = 0$ . Thus,  $\underline{X} \cong \underline{T}_0 \in \underline{\mathcal{T}}$ .  $\square$

Using the above preparation, we now can prove Conjecture 1.1 for all acyclic sign-skew-symmetric cluster algebras. The method of the proof follows from that of Theorem 3.7 (i) in [15].

**Theorem 4.7.** *The conjecture 1.1 on sign-coherence holds for all acyclic sign-skew-symmetric cluster algebras.*

*Proof.* For any cluster  $Z = \{z_1, \dots, z_n\}$  of  $\mathcal{A}(\Sigma)$ , by Theorem 3.6, we associate a cluster tilting subcategory  $\underline{\mathcal{T}}$  of  $\underline{\mathcal{C}}^Q$  which obtained by a series of mutations of  $\underline{\mathcal{T}}_0$  and the mutations do not take at  $\underline{\mathcal{T}}''$ . Precisely, there are  $n$  indecomposable objects  $\{\underline{X}_j \mid j = 1, \dots, n\}$  such that  $\underline{\mathcal{T}} = \text{add}(\{h \cdot \underline{X}_j \mid j = 1, \dots, n\} \cup \underline{\mathcal{T}}'')$  and  $CC'(\underline{X}_i) = z_i$  for  $i = 1, \dots, n$ . By Proposition 4.1, the  $g$ -vector  $g_{[1]}^j, \dots, g_{[n]}^j$  of  $z_j$  is given by  $g_{[i]}^j = \sum_{i' \in [i]} [\text{ind}_{\underline{\mathcal{T}}_0}(\underline{X}_j) : \underline{T}_{i'}]$ .

Suppose that there exist  $s$  and  $s'$  such that  $g_{[i]}^s > 0$  and  $g_{[i]}^{s'} < 0$ . Assume that  $\underline{T}_{-1}^j \rightarrow \underline{T}_0^j \xrightarrow{f^j} \underline{X}_j \rightarrow \underline{T}_1^j[1]$  be the triangle with  $f^j$  is a minimal right  $\underline{\mathcal{T}}_0$ -approximation for  $j = s, s'$ . Thus, there exist  $h, h' \in \Gamma$  such that  $h \cdot \underline{T}_i$  (respectively,  $h' \cdot \underline{T}_i$ ) is a direct summand of  $\underline{T}_0^s$  (respectively,  $\underline{T}_0^{s'}$ ). Furthermore, in the triangle

$$\bigoplus_{j=s, s'} \underline{T}_{-1}^j \rightarrow \bigoplus_{j=s, s'} \underline{T}_0^j \xrightarrow{j=s, s'} \bigoplus_{j=s, s'} \underline{X}_j \rightarrow \bigoplus_{j=s, s'} \underline{T}_1^j[1],$$

the morphism  $\bigoplus_{j=s, s'} f^j$  is a minimal right  $\underline{\mathcal{T}}_0$ -approximation. According to Lemma 4.5,  $h' \cdot \underline{T}_i$  is not a direct summand of  $\underline{T}_0^{s'}$  since  $h \cdot \underline{T}_i$  is a direct summand of  $\underline{T}_0^s$ . This is a contradiction. Our result follows.  $\square$

## 5. THE RECURRENCE OF $\mathbf{g}$ -VECTORS

**Theorem 5.1.** ([6]) *Conjecture 1.3 holds true for all finite rank skew-symmetric cluster algebras.*

It is easy to see that the above theorem can be extended to the situation of infinite rank skew-symmetric cluster algebras, that is, we have:

**Theorem 5.2.** *Conjecture 1.3 holds true for all infinite rank skew-symmetric cluster algebras.*

We first give the following easy lemma.

**Lemma 5.3.** *Let  $(Q, F, \Gamma)$  be the unfolding of a matrix  $B$  and  $\mathcal{A} = \mathcal{A}(\Sigma(Q, F))$ . For any sequence of orbits  $([i_1], \dots, [i_s])$  and  $a \in Q_0$ , there exist finite subsets  $S_j \subseteq [i_j], j = 1, \dots, s$  such that  $\prod_{k \in V_s} \mu_k \cdots \prod_{k \in V_1} \mu_k(x_a) = \tilde{\mu}_{[i_s]} \cdots \tilde{\mu}_{[i_1]}(x_a)$  for all finite subsets  $V_j, j = 1, \dots, s$ , satisfying  $S_j \subseteq V_j \subseteq [i_j], j = 1, \dots, s$ .*

*Proof.* Since  $\tilde{\mu}_{[i_s]} \cdots \tilde{\mu}_{[i_1]}(x_a)$  is determined by finite vertices of  $Q_0$ , there exist finite subsets  $S_j \subseteq [i_j], j = 1, \dots, s$  such that  $\prod_{k \in S_s} \mu_k \cdots \prod_{k \in S_1} \mu_k(x_a) = \tilde{\mu}_{[i_s]} \cdots \tilde{\mu}_{[i_1]}(x_a)$ . Then for all finite subsets  $V_j, j = 1, \dots, s$  satisfying  $S_j \subseteq V_j \subseteq [i_j], j = 1, \dots, s$ , we have  $\prod_{k \in V_s} \mu_k \cdots \prod_{k \in V_1} \mu_k(x_a) = \tilde{\mu}_{[i_s]} \cdots \tilde{\mu}_{[i_1]}(x_a)$ .  $\square$

Let  $\mathcal{A}_1$  (respectively,  $\mathcal{A}_2$ ) be the cluster algebra with principal coefficients at  $\Sigma_1 = (\tilde{X}, \tilde{Y}, Q)$  (respectively,  $\Sigma_2 = (\tilde{X}', \tilde{Y}', \tilde{\mu}_{[k]}(Q))$ ). For any sequence  $([i_1], \dots, [i_s])$  of orbits of  $Q_0$  and  $a \in Q_0 = \tilde{\mu}_{[k]}(Q_0)$ , denote  $g^{Q, a} = (g_i^Q)_{i \in Q_0}$  (respectively,  $g^{\tilde{\mu}_{[k]}(Q), a} = (g_i^{\tilde{\mu}_{[k]}(Q)})_{i \in Q_0}$ ) be the  $\mathbf{g}$ -vector of the cluster variable  $\tilde{\mu}_{[i_s]} \cdots \tilde{\mu}_{[i_1]}(x_a)$  (respectively,  $\tilde{\mu}_{[i_s]} \cdots \tilde{\mu}_{[i_1]} \tilde{\mu}_{[k]}(x'_a)$ ).

As a consequence of Theorem 5.2, we have the following property.

**Proposition 5.4.** *Keep the notations as above. The following recurrence holds:*

$$(3) \quad g_i^{\tilde{\mu}_{[k]}(Q)} = \begin{cases} -g_i^Q & \text{if } i \in [k]; \\ g_i^Q + \sum_{k' \in [k]} [b_{ik'}]_+ g_{k'}^Q - \sum_{k' \in [k]} b_{ik'} \min(g_{k'}^Q, 0) & \text{if } i \notin [k]. \end{cases}$$

*Proof.* By Lemma 5.3, for  $j = 1, \dots, s$ , there exist finite subsets  $S_j^1 \subseteq [i_j]$  (respectively,  $S_j^2 \subseteq [i_j]$ ) such that  $g^{Q,a}$  (respectively,  $g^{\tilde{\mu}_{[k]}(Q),a}$ ) is the  $\mathbf{g}$ -vector of the cluster variable  $\prod_{t \in V_s^1} \mu_t \cdots \prod_{t \in V_1^1} \mu_t(x_a)$  (respectively,  $\prod_{t \in V_s^2} \mu_t \cdots \prod_{t \in V_1^1} \mu_t \tilde{\mu}_{[k]}(x'_a)$ ) for all finite subsets  $V_j^1$  (respectively,  $V_j^2$ ) satisfying that  $S_j^1 \subseteq V_j^1 \subseteq [i_j]$  (respectively,  $S_j^2 \subseteq V_j^2 \subseteq [i_j]$ ). Choose  $S_j = S_j^1 \cup S_j^2$ , we have  $g^{Q,a}$  and  $g^{\tilde{\mu}_{[k]}(Q),a}$  as the  $\mathbf{g}$ -vectors of the cluster variables  $\prod_{t \in S_s} \mu_t \cdots \prod_{t \in S_1} \mu_t(x_a)$  and  $\prod_{t \in S_s} \mu_t \cdots \prod_{t \in S_1} \mu_t \tilde{\mu}_{[k]}(x'_a)$  respectively.

For any finite subset  $T \subseteq [k]$ , denote by  $\mathcal{A}^T$  the cluster algebra with principal coefficients at  $\Sigma^T = (\tilde{X}^T, \tilde{Y}^T, \prod_{k' \in T} \mu_{k'}(Q))$ . Denote by  $g_i^{\prod_{k' \in T} \mu_{k'}(Q),a} = (g_i^{\prod_{k' \in T} \mu_{k'}(Q)})_{i \in Q_0}$  the  $\mathbf{g}$ -vector of the cluster variable  $\prod_{t \in S_s} \mu_t \cdots \prod_{t \in S_1} \mu_t \prod_{k' \in T} \mu_{k'}(x_a^T)$ .

Since the cluster variable  $\prod_{t \in S_s} \mu_t \cdots \prod_{t \in S_1} \mu_t \tilde{\mu}_{[k]}(x'_a)$  is only determined by finite vertices of  $Q_0$ , there exists a finite subset  $S \subseteq [k]$  such that  $g_i^{\tilde{\mu}_{[k]}(Q)} = g_i^{\prod_{k' \in T} \mu_{k'}(Q)}$  for any finite set  $T$  satisfying  $S \subseteq T \subseteq [k]$ .

Furthermore, by Theorem 5.2, for any subset  $T \subseteq Q_0$ , we have

$$(4) \quad g_i^{\prod_{k' \in T} \mu_{k'}(Q)} = \begin{cases} -g_i^Q & \text{if } i \in T; \\ g_i^Q + \sum_{k' \in T} [b_{ik'}] + g_{k'}^Q - \sum_{k' \in T} b_{ik'} \min(g_{k'}^Q, 0) & \text{if } i \notin T. \end{cases}$$

Therefore, the result holds.  $\square$

**Theorem 5.5.** *Conjecture 1.3 holds true for all acyclic sign-skew-symmetric cluster algebras. That is, let  $B = (b_{[i][j]}) \in \text{Mat}_{n \times n}(\mathbb{Z})$  be a sign-skew-symmetric matrix which is mutation equivalent to an acyclic matrix and let  $t_1 \xrightarrow{[k]} t_2 \in \mathbb{T}_n$  and  $B^2 = \mu_{[k]}(B^1)$ . For  $[a] \in \{[1], \dots, [n]\}$  and  $t \in \mathbb{T}_n$ , assume  $\mathbf{g}_{[a];t}^{B^1;t_1} = (g_{[1]}^{t_1}, \dots, g_{[n]}^{t_1})$  and  $\mathbf{g}_{[a];t}^{B^2;t_2} = (g_{[1]}^{t_2}, \dots, g_{[n]}^{t_2})$ , then*

$$(5) \quad g_{[i]}^{t_2} = \begin{cases} -g_{[i]}^{t_1} & \text{if } [i] = [k]; \\ g_{[i]}^{t_1} + [b_{[i][k]}] + g_{[k]}^{t_1} - b_{[i][k]} \min(g_{[k]}^{t_1}, 0) & \text{if } [i] \neq [k]. \end{cases}$$

*Proof.* By Lemma 2.4 and Theorem 2.5, let  $(Q, \Gamma)$  be an unfolding of  $B$ . Assume  $t_1 \xrightarrow{i_1} t'_2 \cdots t'_s \xrightarrow{i_s} t$ . By Theorem 2.9, we have  $g_{[i]}^{t_2} = \sum_{i' \in [i]} g_{i'}^{\tilde{\mu}_{[k]}(Q)}$  and  $g_{[i]}^{t_1} = \sum_{i' \in [i]} g_{i'}^Q$ . By Proposition 4.1 and Lemma 4.5, for  $k', k'' \in [k]$ , both  $g_{k'}^Q$  and  $g_{k''}^Q$  are non-negative or non-positive. Thus,  $\sum_{k' \in [k]} \min(g_{k'}^Q, 0) = \min(\sum_{k' \in [k]} g_{k'}^Q, 0)$ . Using Proposition 5.4, if  $[i] = [k]$ , then

$$g_{[i]}^{t_2} = \sum_{i' \in [i]} g_{i'}^{\tilde{\mu}_{[k]}(Q)} = - \sum_{i' \in [i]} g_{i'}^Q = -g_{[i]}^{t_1};$$

if  $[i] \neq [k]$ , since  $\sum_{k' \in [k]} \min(g_{k'}^Q, 0) = \min(\sum_{k' \in [k]} g_{k'}^Q, 0)$  and  $b_{[i][k]}^{t_1} = \sum_{i' \in [i]} b_{i'k}$ , then

$$g_{[i]}^{t_2} = \sum_{i' \in [i]} (g_{i'}^Q + \sum_{k' \in [k]} [b_{i'k'}^{t_1}] + g_{k'}^Q - \sum_{k' \in [k]} b_{i'k'}^{t_1} \min(g_{k'}^Q, 0)) = g_{[i]}^{t_1} + [b_{[i][k]}^{t_1}] + g_{[k]}^{t_1} - b_{[i][k]}^{t_1} \min(g_{[k]}^{t_1}, 0).$$

The result holds.  $\square$

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